Ruin Probabilities Under Solvency II Constraints

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Abstract

Under Pillar 1 of the Solvency II (SII) directive, the Solvency Capital Requirement (SCR) and MCR (Minimum Capital Requirement) reflect a level of funds that enables insurance (and reinsurance) undertakings to absorb significant losses and give reasonable assurance to policyholders and beneficiaries. In more details, insurance firms are required to guarantee that the SCR coverage ratio stays above a certain level with a large enough probability. Failure to remain above this level MCR HERE AND CURRY ON may trigger regulatory actions to ensure this obligation is fulfilled and the policy holders are protected against insolvency. In this paper, we generalise the classic Poisson risk model to comply with SII regulations (in the above sense). We derive an explicit expression for the 'probability of insolvency' (which is different from the classical ruin probability), in terms of the classic ruin quantities, and establish a relationship between the probability of insolvency and the classic ruin measure. In addition, under the assumption of exponentially distributed claim sizes, we show the probability of insolvency is simply a constant factor of the classic ruin function. Finally, in order to better capture the reality, dividend payments to the companies shareholders are considered and an explicit expression for the probability of insolvency is derived under this modification. Additionally, motivated by the practise. we assume that the shareholders are willing to contrivance the capital injection tool if the claim amounts force

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1 Introduction

Solvency II is the new harmonised EU regulatory directive for insurance firms, implemented from January 2016 (Directive 2009/138/EC, see [1]). The new regulatory regime introduces capital requirements (based on a prospective risk approach), under which the policyholders protection (security) is improved, the firms can adopt better risk management strategies (by direct the capital accurately where the risks are), while the prudential authorities and EIOPA (European Insurance and Occupational Pensions Authority) can monitor effectively the insurance institutions (under a modernised supervision scheme). The Solvency II framework consists of three pillars. Pillar 1 comprises the quantitative capital requirements, Pillar 2 comprises the risk management quality requirements, while Pillar 3 comprises the regulator supervisory and public disclosure requirements.

In practise, within Pillar 1, actuaries apply the so called standard formula or internal models in order match the assets with the future and current liabilities and eventually to evaluate and assess the capital requirements of insurance firms. In more details, Pillar 1 sets an upper and a lower level of capital requirement, in which in the first case the insurance firm is considered to be sufficiently capitalised, while the latter triggers the supervisory intervention due to insufficient capital holding. The aforementioned upper level is called Solvency Capital Requirement (SCR) and has to be fulfilled by insurance institutions to assure a theoretical ruin probability of 0.005 (this ensures that ruin occurs no more often than once in every 200 years). The Minimum Capital Requirement (MCR) is the level below which the regulator's strongest actions are taken (e.g. recovery plan requirement or removal of the insurer's authorisation). The MCR is calculated (usually) using a linear formula and must fall between 25% and 45% of the SCR.

The basic underlying assumption within SII regulation is that SCR is calibrated using the Value at Risk (VaR) of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5 % over a one-year period. This calibration is applied to each individual risk module and sub-module of all risks that an insurance firm faces. The same kind of assumption lies in the heart of regulatory regimes for capital requirements that are applied in the US (Risk Base Capital, RBC, see [2]), in China (China Risk Oriented Solvency System, C-ROSS, see [3]), or Switzerland (Swiss Solvency Test, see [10]). The strong connection between the VaR and the ruin probability has been studied by Trufin et al [4], Ren [5], Gerber and Loisel [6], Gatto and Baumgartner [7] and there references therein. As pointed out in Gerber and Loisel [6], ruin theory provides a more sustainable valuation principle (than the single use of the VaR approach) since it takes into account liquidity constraints and penalises large position sizes.

The risk process we employee to model the SII framework consists of the following characteristics:

a. We consider a compound Poisson risk process for which two barriers are employed to model the MCR and the SCR level. We assume that the insurance firm starts from a solvent level which exceeds the SCR level and has downward jumps due to the claim arrivals of the Poisson process. Once, the SCR level has been crossed, due to a claim, then the insurance firm has to recover the capital so as to meet the SCR level again, and hence to fulfil the SII capital requirements which indicate specific values for the SCR level of an insurance firm.

b. Following Solvency II and market studies, we consider in our model that the aforementioned recover in terms of capital could be provided by capital injections, given the MCR level has not been crossed by the claim amounts (see also Section 2 and Figure 1). The capital injection is a re-capitalisation mechanism often implemented under the SII environment, see for example, among others, the case of the ING group insurance in Netherlands (see [8]), the case of Liberty Insurance in Ireland (see [9]), or MOODY'S report of April 2016.

c. Additionally, motivated by again by the practise, we assume that there exists an intermediate capital level barrier, in between SCR and MCR, which indicates the confidence level of which the share holders are prepared to inject capital in order the surplus to be restored back to the SCR level. If the claim appears to be large enough such this intermediate confidence level is crossed, then the recovery actions of the insurance firm is to borrow capital at a debit interest rate until the intermediate confidence level of the share holders will be reached again and hence the SCR level can be restored again by a capital injection.

d. Further, during the borrowing period if another claims occurs, causing the risk process to drop to the MCR level or further, then the firm cannot longer considered as solvent and thus the regulatory worst actions have to take place.

e. We underline that if a claims occurs, which lead to the drop of the risk process to the MCR level directly, then the regulatory actions are immediately in effect.

Capital injections have been first introduced in the risk theory context by Parfumi (1998). The ruin probability and other ruin related quantities, such as the distribution of the deficit at ruin or the distribution of the surplus prior to ruin, have been extensively studied for the compound Poisson risk model by many authors, see among others, Nie et al. (2011), Eisenberg and Schmidli (2011), Dickson and Qazvini (2016) and the references therein. The debit interest risk model was first introduced by Dickson and Dos Reis (1997). Explicit expressions for the absolute ruin probabilities and other ruin related quantities have been derived, for the classical risk model, by Cai (2007), Yang and Zhu (2008), Li and Lu (2013) and the references therein. Although that SII regulation is the framework under which insurance firms are nowadays operating, it appears that only a few papers have been written in the risk theory context. Ferriero (2016) derives practical estimators for the capital requirements in a fractional brownian motion risk model. Floryszczak et al. (2016) confirm that the least-squares Monte Carlo method is relevant to SII framework for the capital requirements of an insurance firm. Asimit et al. (2015) propose optimal allocations for the premium and the liabilities in order the MCR level to be reduced.

In this paper we employ the aforementioned SII risk model to study the probability of insolvency. In more details, we show that insolvent probability under the above SII environment can be evaluated in terms of the ruin probability of the classical risk model, for which powerful methodologies, numerical techniques and many applicable results have been derived over the last half century. Additionally, we derive the distribution of the capital injection up to the time that the firm runs off.

The paper is organised as follows:

2 The SII Risk Model

In this section we will adapt the classical risk model to conform with the SII regulatory framework, in order to establish a construction for the SII risk model.

In the classical Cramér-Lundberg risk model, the surplus process of an insurance company is defined by eqC

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geqslant 0, \tag{2.1}$$

where $u \geqslant 0$ is the insurer's initial capital, $c > 0$ is a constant representing the continuously received premium rate, $\{N(t)\}_{t\geqslant 0}$ is a Poisson process denoting the number of claims that have arrived up to time $t \geqslant 0$, with intensity $\lambda > 0$, and $\{X_k : k \in \mathbb{Z}_+\}$ is a sequence of independent and identically distributed (i.i.d) claim size random variables with a common distribution function $F_X(\cdot)$, density $f_X(\cdot)$, and mean $\mathbb{E}(X) = \mu < \infty$. We further assume that $\{N(t)\}_{t\geqslant 0}$ and $\{X_k : k \in \mathbb{Z}_+\}$ are mutually independent.

In practise an insurance company needs, and are obligated under the SII directive, to hold a certain MCR level of capital (which depends on their risk) in order to continue operating. If the surplus of the insurance firm falls below this certain MCR level, then 'ultimate supervisory action' will be triggered. That is, the company could be liquidated, its liabilities could be transferred to another company and its license could be withdrawn. Therefore, in reality, the level of ruin for an insurance firm is much higher than that of zero (as is seen in the classic ruin set up). Under this consideration we will define the 'insolvency probabilities' corresponding to the probabilities that the surplus process down-crosses a certain lower level of capital, namely the MCR.

Note that, although in the SII directive the one year VaR at a 99.5% is used to determine the SCR level, in this paper we focus on the (ruin) insolvency probabilities. The strong connection between VaR and ruin probabilities has been studied in Ren (2012), Denis et al. (2009) and references therein. An additional reason for focusing on the study of infinite time insolvency probabilities is that, in the sequel, we will establish a closed form relation between the insolvency probabilities and the ruin probability of the classical risk model, for which numerous results exist in the Actuarial literature.

Motivated by the Solvency II (SII) directive (Directive 2009/138/EC of the European Parliament and of the Council), we will consider capital injections - which often appear in practise - and borrowing actions that the insurer may consider as a means of maintaining an appropriate level of capital/ SCR level. There are several aspects to the directive that all play important roles in its implication, however, for the purpose of this paper we are going to concentrate on the calculation of the reserves and consequently the insolvency probability.

We assume that if the surplus of the insurer, as defined in equation (2.1), falls below the $SCR\,(\equiv k)$ barrier then the stake holders in the company will inject capital instantaneously to cover this fall. That is, if the surplus falls below the barrier $k \geqslant 0$, by some amount $x > 0$, then there is an instantaneous jump, of size x, back to the SCR level. The sum of total capital injections, up to time $t \geqslant 0$, is defined by the pure jump process $\{Z(t)\}_{t\geqslant 0}$.

In addition, there is an extra precaution if the surplus of the insurer falls below a lower barrier, $k \geqslant b \geqslant 0$. When the surplus drops below this level, the stake holders can no longer afford to inject capital into the company and instead the company must borrow an amount of money equal to the size of the deficit below b continuously, at a debit force $\delta > 0$.

Meanwhile, the insurer will repay the debts continuously from its premium income. The surplus process may return to the level b, at which point the stakeholders have renewed confidence and will inject the amount $k - b$ in order for the process to jump back to level k. However, if the surplus ever falls below the $MCR\,(\equiv \tilde{b})$ level, the surplus is no longer able to return to the level b, therefore the company becomes 'insolvent' and has to be liquidated. By similar arguments as in Cai (2007) it is easy to see that $b = \tilde{b} + \frac{c}{\delta}$ since, at the point $b - c/\delta = \tilde{b}$, the debts of the insurer are greater than the present value for all premium income available after that point. Insolvency occurs at this point.

Note that all the aforementioned features are strongly connected to the capital level that an insurer must hold during its operating time and thus is strongly correlated with SII.

Introducing these features, the amended surplus process, denoted by $\{U_\delta^Z(t)\}_{t\geqslant 0}$, has dynamics eqDynam

$$dU_\delta^Z(t) = \begin{cases} cdt - dS(t), & U_\delta^Z(t) \geqslant k, \\ \Delta Z(t), & b \leqslant U_\delta^Z(t) < k, \\ \left[c + \delta(U_\delta^Z(t) - b)\right] dt - dS(t), & \tilde{b} < U_\delta^Z(t) < b, \end{cases} \tag{2.2}$$

where $\Delta Z(t) = Z(t) - Z(t-)$ and $S(t) = \sum_{i=1}^{N(t)} X_i$.

Within this new legislation there are rules that stipulate the minimum reserves an insurance company must hold in order to cover their exposed risks, and so it follows that for the surplus process $\{U_\delta^Z(t)\}_{t\geqslant 0}$, we should define the time to insolvency, denoted by T_δ, as

$$T_\delta = \inf\left\{t \geqslant 0 : U_\delta^Z(t) \leqslant \tilde{b} | U_\delta^Z(0) = u\right\},$$

with $T_\delta = \infty$ if $U_\delta^Z(t) > \tilde{b}$ for all $t \geqslant 0$. Then, the probability of insolvency (ruin) will be denoted by $\psi_{\mathrm{SII}}(u)$, and is given by

$$\psi_{\mathrm{SII}}(u) = \mathbb{P}\left(T_\delta < \infty \middle| U_\delta^Z(0) = u\right),$$

with $\psi_{\mathrm{SII}}(u) = 1$ for $u \leqslant \tilde{b}$ and $\phi_{\mathrm{SII}}(u) = 1 - \psi_{\mathrm{SII}}(u)$ being the probability of solvency (survival).

Justify the finite time versus infinite that we study in SECTION 2

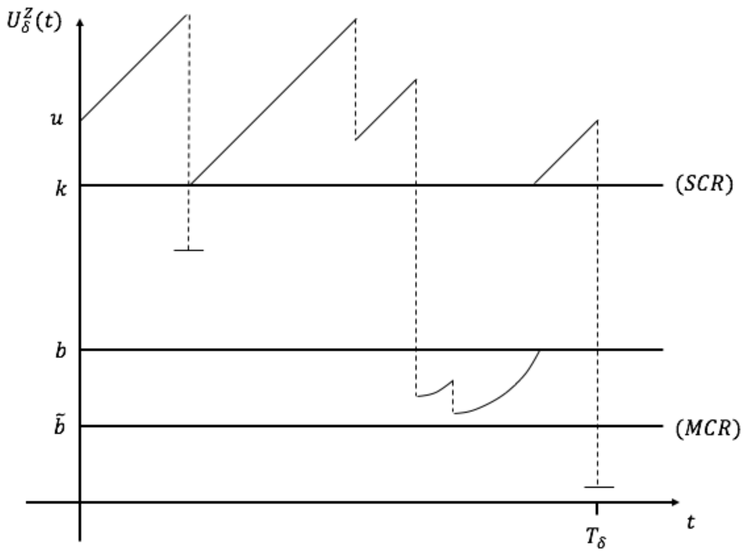

Figure 1: Example sample path of the surplus process under SII constraints

We point out, similar to Cai (2007), that $\psi_{\mathrm{SII}}(u)$ has different sample paths for $u \geqslant k$ and $\tilde{b} < u < b$. Therefore, we distinguish the two situations by writing $\psi_{\mathrm{SII}}(u) = \psi_{\mathrm{SII}}^+(u)$ for $u \geqslant k$ and $\psi_{\mathrm{SII}}(u) = \psi_{\mathrm{SII}}^-(u)$ for $\tilde{b} < u < b$. Now, due to the instantaneous capital injection when the surplus lies within the interval $[b, k)$ we say that for $b \leqslant u < k$, $\psi_{\mathrm{SII}}(u) = \psi_{\mathrm{SII}}^+(k)$. It follows that the corresponding solvency probabilities are given by $\phi_{\mathrm{SII}}(u) = 1 - \psi_{\mathrm{SII}}(u) = \phi_{\mathrm{SII}}^+(u)$, for $u \geqslant k$, and $\phi_{\mathrm{SII}}(u) = \phi_{\mathrm{SII}}^-(u)$ for $\tilde{b} < u < b$. Finally, we assume the net profit condition holds, that is eqnetprof

$$\eta = (c/\lambda\mu) - 1 > 0. \tag{2.3}$$

3 Ruin probabilities under SII model

In this section, we derive a closed form expression for the probability of insolvency $\psi_{\mathrm{SII}}^+(u)$, $u \geqslant k$, in terms of the ruin probability of the classical risk model and an exiting (hitting) probability between two barriers. Ultimately, we will show that the probability of insolvency is given as a proportion of the 'shifted' classical ruin function. We will also derive, out of mathematical curiosity (since Solvency II regulation stipulates initial capital must exceed the SCR level), corresponding formulae for the $\psi^-_{\text{SII}}(u)$, $\tilde{b} < u < b$.

Before we proceed, let us first remind the reader of some ruin related quantities that will be extensively used in the following. First, let the time to cross the barrier k, for $u \geqslant k$, be denoted by T, such that eqcrossT

$$T = \inf\{t \geqslant 0 : U^Z_\delta(t) < k |\, U^Z_\delta(0) = u \geqslant k\}. \tag{3.1}$$

Then, we are able to define the probability of such an event occurring, i.e. the probability of down crossing the barrier k, by

$$\xi(u) = \mathbb{P}\left(T < \infty \middle| U^Z_\delta(0) = u \geqslant k\right).$$

Recalling the behaviour of the surplus process $U^Z_\delta(t)$ given in equation (2.2), it is clear to see that the dynamics above the barrier k are identical to that of the classical surplus process under a free barrier environment, i.e. for $u \geqslant k$, we have $dU^Z_\delta(t) \equiv d\widetilde{U}(t)$ where

$$\widetilde{U}(t) = \tilde{u} + ct - S(t), \quad t \geqslant 0,$$

with $\widetilde{U}(0) = \tilde{u} = u - k$. It should then be clear to see that T, defined by equation (3.1), is equivalent to the *time to ruin* in the classical risk model with no barrier modification and initial capital $\tilde{u} \geqslant 0$, given by eqCRT

$$T = \inf\{t \geqslant 0 : \widetilde{U}(t) < 0 |\, \widetilde{U}(0) = \tilde{u}\}, \tag{3.2}$$

and that the function $\xi(u)$ is identical to the classic ruin probability $\psi(\tilde{u}) = \mathbb{P}(T < \infty | \widetilde{U}(0) = \tilde{u})$. Moreover, the probability of never crossing the barrier k can be expressed by the classic survival probability $\phi(\tilde{u}) = 1 - \psi(\tilde{u})$.

Now that we have made apparent the equivalence between the distribution of crossing the k barrier with classical ruin, let us define

$$G(\tilde{u}, y) = \mathbb{P}\left(T < \infty, |\widetilde{U}(T)| \leqslant y \middle| \widetilde{U}(0) = \tilde{u}\right),$$

as the joint distribution of crossing below the barrier k and experiencing a drop of at most y, with $g(\tilde{u}, y) = \frac{\partial}{\partial y} G(\tilde{u}, y)$ the corresponding density function. This quantity is equivalent to the joint distribution introduced by Gerber et al. (1987) for the 'deficit at ruin'.

For the ease of calculations, the results in the following will be derived initially in terms of the solvency probabilities $\phi^+_{\text{SII}}(u)$ and $\phi^-_{\text{SII}}(u)$, for $u \geqslant k$ and $\tilde{b} < u < b$ respectively.

Extending an argument of Nie et al. (2011), by conditioning on the occurrence and size of the first drop below k, for $u \geqslant k$, we obtain the following expression for the solvency probability eqCR1

$$\phi_{\mathrm{SII}}^{+}(u) = \phi(\tilde{u}) + \int_{0}^{k-b} g(\tilde{u},y)\phi_{\mathrm{SII}}^{+}(k)\,dy + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},y)\phi_{\mathrm{SII}}^{-}(k-y)\,dy$$
$$= \phi(\tilde{u}) + G(\tilde{u},k-b)\phi_{\mathrm{SII}}^{+}(k) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},y)\phi_{\mathrm{SII}}^{-}(k-y)\,dy. \tag{3.3}$$

In order to simplify the above into a more tractable equation, we want to express the solvency function $\phi_{\mathrm{SII}}^{-}(u)$ in terms of $\phi_{\mathrm{SII}}^{+}(u)$. This can be done by the introduction of a exiting (hitting) probability.

Consider the time T^b of hitting an upper barrier b, given the surplus starts with initial capital $b > u > \tilde{b}$. Then, we are able to express the exiting (hitting) probability function $\chi_\delta(u,b,\tilde{b}) \equiv \chi_\delta(u)$, representing the probability of hitting the upper barrier b before hitting the lower barrier $\tilde{b}$ under the debit force, by eqChi1

$$\chi_\delta(u) = \mathbb{P}\left(T^b < T_\delta \middle| U_\delta^Z(0) = u\right), \tag{3.4}$$

where

$$T^b = \inf\left\{t \geqslant 0 : U_\delta^Z(t) = b \middle| U_\delta^Z(0) = u\right\}, \quad \tilde{b} < u < b.$$

If we consider a conditioning argument on the possible events, starting from initial capital $\tilde{b} < u < b$, then, noting that $\phi_{\mathrm{SII}}^{-}(x) = 0$ for $x \leqslant \tilde{b}$, and recalling the definition of the exiting probability defined in equation (3.4), it follows from the law of total probability that eqMin

$$\phi_{\mathrm{SII}}^{-}(u) = \chi_\delta(u)\phi_{\mathrm{SII}}^{+}(k), \tag{3.5}$$

from which, after substituting into equation (3.3), we obtain eqCR2

$$\phi_{\mathrm{SII}}^{+}(u) = \phi(\tilde{u}) + \phi_{\mathrm{SII}}^{+}(k)\left[G(\tilde{u},k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},y)\chi_\delta(k-y)\,dy\right]. \tag{3.6}$$

If we consider the case $u = k$, it allows us to solve the above equation with respect to $\phi_{\mathrm{SII}}^{+}(k)$, from which we acquire an explicit expression of the form eqCR3

$$\phi_{\mathrm{SII}}^{+}(k) = \frac{\phi(0)}{1 - \left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy\right)}. \tag{3.7}$$

Finally, by combining equations (3.6) and (3.7), we are able to formulate an expression for the solvency probability, for $u \geqslant k$, given by eqCR4

$$\phi_{\mathrm{SII}}^{+}(u) = \phi(\tilde{u}) + \frac{\phi(0)\left[G(\tilde{u},k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},y)\chi_\delta(k-y)\,dy\right]}{1 - \left(G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy\right)}, \tag{3.8}$$

where $\tilde{u} = u - k$ or equivalently, for the insolvency (ruin) probability, by eqCR8

$$\psi^{+}_{\mathrm{SII}}(u) = \psi(\tilde{u}) - \frac{\phi(0)\left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_{\delta}(k-y)\,dy\right]}{1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_{\delta}(k-y)\,dy\right)}. \tag{3.9}$$

Remark 1. *Note that the numerator in equation* (3.9) *comprises of probability functions and thus is clearly positive. Further, by dominated convergence theorem we have*

$$\begin{aligned}\int_{k-b}^{k-\tilde{b}} g(0, y)\chi_{\delta}(k-y)\,dy &\leqslant \int_{k-b}^{k-\tilde{b}} g(0, y)\,dy \\ &= G(0, k-\tilde{b}) - G(0, k-b),\end{aligned}$$

and it follows that

$$\begin{aligned}1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_{\delta}(k-y)\,dy\right) &\geqslant 1 - \left(G(0, k-b) + G(0, k-\tilde{b}) - G(0, k-b)\right) \\ &= 1 - G(0, k-\tilde{b}) > 0,\end{aligned}$$

by the net profit condition. Therefore, the fraction on the right hand side of equation (3.9) *is positive and the probability of insolvency, for $u \geqslant k$, is less than the shifted classical ruin probability.*

From equation (3.9), it should be clear that the probability of insolvency, namely $\psi^{+}_{\mathrm{SII}}(u)$, heavily depends on the distribution function of the deficit at ruin of the classical risk model. Then, using from Dickson (2005) the fact that the general form for the density of the deficit at ruin (with zero initial capital) is simply a proportion of the tail distribution i.e.

$$g(0, y) = \frac{\lambda}{c}\overline{F}_X(y),$$

equation (3.9) reduces to eqCR5

$$\psi^{+}_{\mathrm{SII}}(u) = \psi(\tilde{u}) - \frac{\phi(0)\left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_{\delta}(k-y)\,dy\right]}{1 - \frac{\lambda}{c}\left(\mu F_e(k-b) + \int_{k-b}^{k-\tilde{b}} \overline{F}_X(y)\chi_{\delta}(k-y)\,dy\right)}, \tag{3.10}$$

where $G(0, y) = \int_0^y g(0, z)\,dz$, $\overline{F}_X(x) = 1 - F_X(x)$ and $F_e(x) = \frac{1}{\mu}\int_0^x \overline{F}_X(y)\,dy$ is the so-called equilibrium distribution.

Finally, by employing equation (3.10), combining equations (3.5) and (3.7) and defining $G_{\tilde{u}}(y) = \frac{G(\tilde{u}, y)}{\psi(\tilde{u})}$, with $g_{\tilde{u}}(y) = \frac{g(\tilde{u}, y)}{\psi(\tilde{u})}$, such that $G_{\tilde{u}}(y) = \mathbb{P}(|\widetilde{U}(T)| \leqslant y \big| T < \infty)$ is a proper distribution function, as in Willmot (2002) (and references therein), we get the following Theorem for the probability of insolvency. ThmS1

Theorem 1. *For $u \geqslant k$, the probability of insolvency, $\psi^{+}_{SII}(u)$, is given by* eqCRL6

$$\psi^{+}_{SII}(u) = \psi(\tilde{u}) \left[1 - \frac{\phi(0) \left[G_{\tilde{u}}(k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{u}}(y) \chi_{\delta}(k-y)\, dy \right]}{1 - \frac{\lambda}{c} \left(\mu F_e(k-b) + \int_{k-b}^{k-\tilde{b}} \overline{F}_X(y) \chi_{\delta}(k-y)\, dy \right)} \right], \tag{3.11}$$

where $\psi(\tilde{u}) = \psi(u-k)$ is the shifted classical ruin function and for $\tilde{b} < u < b$, $\psi^{-}_{SII}(u)$, we have eqCRLm

$$\psi^{-}_{SII}(u) = 1 - \frac{\phi(0) \chi_{\delta}(u)}{1 - \frac{\lambda}{c} \left(\mu F_e(k-b) + \int_{k-b}^{k-\tilde{b}} \overline{F}_X(y) \chi_{\delta}(k-y)\, dy \right)}. \tag{3.12}$$

Rem2

Remark 2. *From equations (3.11) and (3.12), it follows that the two types of insolvency probabilities are given in terms of the (shifted) ruin probability and deficit of the classical risk model, as well as the probability of exiting between two barriers. Thus, $\psi^{+}_{SII}(\cdot)$ and $\psi^{-}_{SII}(\cdot)$ can be calculated by employing the well known results, with respect to $G_{\tilde{u}}(\cdot)$ and $\psi(\cdot)$ (see for example Gerber et al. (1987), Dickson (2005), and the references therein), whilst the latter exiting probability, $\chi_{\delta}(u)$, is calculated as follows.*

Following similar arguments of Cai (2007), we get the following Proposition. PropC1

Proposition 1. *For $\tilde{b} < u < b$, the probability of hitting an upper barrier b before hitting a lower barrier $\tilde{b}$ (under a debit environment), denoted $\chi_{\delta}(u)$, satisfies the following integro-differential equation* eqchi

$$(\delta(u-b)+c)\chi'_{\delta}(u) = \lambda \chi_{\delta}(u) - \lambda \int_0^{u-\tilde{b}} \chi_{\delta}(u-x)\, dF_X(x), \tag{3.13}$$

with boundary conditions

$$\lim_{u \uparrow b} \chi_{\delta}(u) = 1,$$
$$\lim_{u \downarrow \tilde{b}} \chi_{\delta}(u) = 0.$$

Proof. Let us first note that when the surplus process is within the interval $(\tilde{b}, b)$, it is driven by the debit interest force $\delta > 0$, until the surplus returns to level b (or experiences insolvency). Therefore, for initial capital $\tilde{b} < u < b$, the process is immediately subject to debit interest on the amount $b - u > 0$ and the evolution of the surplus process (assuming no claims appear up to time $t \geqslant 0$), can be expressed by eqh

$$h(t,u,b) = b + (u-b)e^{\delta t} + c\int_0^t e^{\delta s}\, ds, \qquad t \geqslant 0.$$

Now, let us further define $t_0 \equiv t_0(u,b)$ to be the solution to $h(t,u,b) = b$, where eqt

$$t_0 = \ln\left(\frac{c}{\delta(u-b)+c}\right)^{1/\delta}, \tag{3.14}$$

is the time taken for the surplus to reach the upper barrier level b i.e. $h(t_0,u,b) = b$, in the absence of claims and $h(t,u,b) \in (\tilde{b}, b)$ for all $t < t_0$. Then, by conditioning on the time and amount of the first claim, it follows that eqCHI11

$$\chi_\delta(u) = e^{-\lambda t_0} + \int_0^{t_0} \lambda e^{-\lambda t} \int_0^{h(u,t,b)-\tilde{b}} \chi_\delta\Big(h(u,t,b) - x\Big)\, dF_X(x)\, dt. \tag{3.15}$$

Using the change of variable $y = h(t,u,b)$, we obtain that eqCHI2

$$\chi_\delta(u) = \left(\frac{\delta(u-b)+c}{c}\right)^{\frac{\lambda}{\delta}} + \lambda\left(\delta(u-b)+c\right)^{\frac{\lambda}{\delta}} \int_u^b \left(\delta(y-b)+c\right)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF_X(x)\, dy. \tag{3.16}$$

Differentiating the above equation, with respect to u, and combining the resulting equation with equation (3.15), we obtain equation (3.13).

The first boundary condition is found by letting $u \to b$ in equation (3.16). Now, for the second boundary condition one can see that if

$$\lim_{u \downarrow \tilde{b}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF(x)\right] dy < \infty,$$

then

$$\lim_{u \downarrow \tilde{b}} \lambda\big(\delta(u-b)+c\big)^{\frac{\lambda}{\delta}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF(x)\right] dy = 0,$$

since $\tilde{b} = b - \frac{c}{\delta}$. Alternatively, if

$$\lim_{u \downarrow \tilde{b}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF(x)\right] dy = \infty,$$

then, by L'Hopital's rule, we have

$$\lim_{u \downarrow \tilde{b}} \lambda\big(\delta(u-b)+c\big)^{\frac{\lambda}{\delta}} \int_u^b \left[\big(\delta(y-b)+c\big)^{-\frac{\lambda}{\delta}-1} \int_0^{y-\tilde{b}} \chi_\delta(y-x)\, dF(x)\right] dy = 0.$$

Using the above limiting results and taking the limit $u \to \tilde{b}$, in equation (3.16), we obtain the second boundary condition. $\square$

Remark 3. *We point out that the integral form of equation* (3.15) *allows us to consider the differentiability of* $\chi_\delta(u)$, $\tilde{b} < u < b$.

Recalling Remark 2 and Theorem 1, the two types of insolvency probabilities depend heavily on the solution of the integro-differential equation (3.13), which is discussed in the next subsection.

3.1 Explicit expression for exponential claim size distribution

In this subsection, we derive explicit expressions for the two types of insolvency probabilities, under the assumption of exponentially distributed claim amounts, by calculating first $\chi_\delta(u)$ with exponential claims. Then, by comparing the explicit expression of the insolvency probabilities with the classical ruin probability, we identify that the probability of insolvency is given as a constant proportion of the probability of ruin in the classical model. To illustrate the applicability of our results (and thus the relation between $\psi^{+}_{\mathrm{SII}}(u)$ and $\psi(u)$), we finally provide numerical results.

Let us assume the claim sizes are exponentially distributed with parameter $\beta > 0$ i.e. $F_X(x) = 1 - e^{-\beta x}$, $x \geqslant 0$. Then, equation (3.13) reduces to eqExp

$$(\delta(u-b)+c)\chi'_\delta(u) = \lambda\chi_\delta(u) - \lambda\int_{\tilde{b}}^{u} \beta e^{-\beta(u-x)}\chi_\delta(x)\,dx, \qquad \tilde{b} < u < b. \tag{3.17}$$

The above integro-differential equation can be solved as a boundary value problem, since from Proposition 1 the boundary conditions at $\tilde{b}$ and b are given. Thus, differentiating the above equation with respect to u, it yields a second order homogeneous ODE of the form

$$(\delta(u-b)+c)\chi''_\delta(u) + (\delta - \lambda + \beta[\delta(u-b)+c])\chi'_\delta(u) = 0,$$

or equivalently eqExp1

$$\chi''_\delta(u) + p(u)\chi'_\delta(u) = 0, \tag{3.18}$$

where

$$p(u) = \frac{\delta - \lambda + \beta[\delta(u-b)+c]}{\delta(u-b)+c} = \frac{\delta-\lambda}{\delta(u-b)+c} + \beta.$$

The above equation can now be solved by employing the general theory of differential equations, as follows. Let us define the auxiliary function $g(u) = \chi'_\delta(u)$, for $\tilde{b} < u < b$. Then, equation (3.18) reduces to

$$g'(u) + p(u)g(u) = 0,$$

which has a general solution of the form

$$g(u) = Ce^{-\int p(u)\,du},$$

where C is an arbitrary constant that needs to be determined in order to complete the above solution. Recalling the form of $p(u)$, the general solution of the above ODE is given by

$$g(u) = Ce^{-\beta u} \left(\delta(u-b)+c\right)^{\frac{\lambda}{\delta}-1}.$$

Now, integrating the above equation from $\tilde{b}+\epsilon$ to u, and since $g(u) = \chi'_\delta(u)$, we have that

$$\chi_\delta(u) - \chi_\delta(\tilde{b}+\epsilon) = C\int_{\tilde{b}+\epsilon}^{u} e^{-\beta w} \left(\delta(w-b)+c\right)^{\frac{\lambda}{\delta}-1} dw.$$

Letting $\epsilon \to 0$ and using the second boundary condition of Proposition 1, the general solution of equation (3.18) is given by eqCHI1

$$\begin{aligned}\chi_\delta(u) &= C\int_{\tilde{b}}^{u} e^{-\beta w} \left(\delta(w-b)+c\right)^{\frac{\lambda}{\delta}-1} dw \\ &= Cc^{\frac{\lambda}{\delta}-1}\int_{\tilde{b}}^{u} e^{-\beta w} \left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1} dw. \qquad (3.19)\end{aligned}$$

In order to complete the solution, we need to determine the constant C, which can be obtained by using the second boundary condition for $\chi_\delta(u)$ of Proposition 1 i.e. $\lim_{u\to b}\chi_\delta(u) = 1$. That is, by letting $u \to b$ in equation (3.19), we obtain

$$\begin{aligned}C^{-1} &= c^{\frac{\lambda}{\delta}-1}\int_{\tilde{b}}^{b} e^{-\beta w} \left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1} dw \\ &= c^{\frac{\lambda}{\delta}-1}C_1^{-1},\end{aligned}$$

where $C_1^{-1} = \int_{\tilde{b}}^{b} e^{-\beta w}\left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1} dw.$ PropC2

Proposition 2. *For $\tilde{b} < u < b$, and exponentially distributed claim amounts with parameter $\beta > 0$, the probability of hitting the upper barrier b, before hitting the lower barrier $\tilde{b}$, under a debit environment, is given by* eqC1

$$\chi_\delta(u) = C_1\int_{\tilde{b}}^{u} e^{-\beta w} \left(\frac{\delta(w-b)}{c}+1\right)^{\frac{\lambda}{\delta}-1} dw. \qquad (3.20)$$

Using Theorem 1 and Proposition 2, the two types of insolvency probabilities, namely $\psi^+_{\mathrm{SII}}(u)$ and $\psi^-_{\mathrm{SII}}(u)$, under exponentially distributed claim amounts, are given in the following Theorem.

Theorem 2. *Let the claim amounts be exponentially distributed with parameter $\beta > 0$. Then, the two types of insolvency probabilities are given by, for $u \geqslant k$;* eqPSI

$$\psi^+_{SII}(u) = \frac{(1+\eta)e^{\frac{\lambda\eta}{c}k}}{1+\frac{\lambda\eta}{c}C_1^{-1}e^{\beta k}}\,\psi(u), \qquad (3.21)$$

and, for $\tilde{b} < u < b$; eqPSI2

$$\psi^{-}_{SII}(u) = \frac{\left(1 - C_1 \int_{\tilde{b}}^{u} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw\right)\eta + C_1 \frac{c}{\lambda} e^{-\beta k}}{\eta + C_1 \frac{c}{\lambda} e^{-\beta k}}, \tag{3.22}$$

where eqConst

$$C_1^{-1} = \int_{\tilde{b}}^{b} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw. \tag{3.23}$$

Proof. Let us begin by considering the numerator in equation (3.11), given by

$$\phi(0)\left[G_{\tilde{u}}(k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{u}}(y)\chi_{\delta}(k-y, b, \tilde{b})\, dy\right].$$

Assuming that the claim amounts are exponentially distributed, employing the corresponding forms for $G_{\tilde{u}}(y)$ and $g_{\tilde{u}}(y)$ from Dickson(2005) and using equation (3.20) of Proposition 2, it follows that the above equation may be written as

$$\phi(0)\left[\left(1 - e^{-\beta(k-b)}\right) + C_1\beta \int_{k-b}^{k-\tilde{b}} e^{-\beta y} \int_{\tilde{b}}^{k-y} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dwdy\right].$$

Changing the order of integration, evaluating the resulting inner integral and applying some algebraic manipulations, we obtain that

$$\phi(0)\left[1 - e^{-\beta(k-b)}\left(1 - C_1 \int_{\tilde{b}}^{b} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw\right) - C_1 \frac{c}{\lambda} e^{-\beta k}\right].$$

Furthermore, recalling the definition of the constant C_1, given in equation (3.23), the above equation reduces to the concise form

$$\phi(0)\left[1 - C_1 \frac{c}{\lambda} e^{-\beta k}\right].$$

Now, considering a similar methodology as above, the corresponding denominator in equation (3.11) reduces to the form

$$1 - \frac{1}{1+\eta}\left(1 - C_1 \frac{c}{\lambda} e^{-\beta k}\right).$$

Finally, substituting the above forms of the numerator and denominator of equation (3.11), we have that the insolvency probability, for $u \geqslant k$, is given by

$$\psi^{+}_{\mathrm{SII}}(u) = \psi(\tilde{u})\left(1 - \frac{\phi(0)A}{1 - \frac{1}{1+\eta}A}\right),$$

where

$$A = \left(1 - C_1 \frac{c}{\lambda} e^{-\beta k}\right).$$

Finally, re-arranging the above equation, substituting the forms of both $\phi(0)$ and $\psi(\tilde{u})$, under exponentially distributed claim sizes (see Grandell (1991)) and noticing that $\psi(\tilde{u}) = \psi(u-k) = e^{\frac{\lambda\eta}{c}k}\psi(u)$, we obtain our result. For $\psi^{-}_{\mathrm{SII}}(u)$, given by equation (3.22), we follow similar arguments and thus the proof is omitted. □

Remark 4. *(i) From equation* (3.21)*, we conclude that the function* $\frac{(1+\eta)e^{\frac{\lambda\eta}{c}k}}{1+\frac{\lambda\eta}{c}C_1^{-1}e^{\beta k}}$ *plays the role of a measurement of protection' for the insurer. By this we mean that given a set of parameters, the above factor could lead to lower (higher) value of* $\psi^{+}_{SII}(u)$ *in the sense that the insurer is more (less) protected by the SII regulations compared with the classical ruin risk measure.*

(ii) In practise insurance firms per-determine their insolvency probability (or equivalent VaR measure), usually at 0.05%. Since equation (3.21) *can be also be written as* eqPSI5

$$\psi^{+}_{SII}(u) = \frac{1}{1 + \frac{\lambda\eta}{c}C_1^{-1}e^{\beta k}} e^{-\frac{\lambda\eta}{c}(u-k)}, \tag{3.24}$$

it follows that, for a fixed value of $\psi^{+}_{SII}(u)$ *and given set of parameters (including the initial capital), we can obtain the required SCR level* k *by solving equation* (3.24) *with respect to* k*.*

Remark 5. *If we set* $k = b = 0$ *such that* $\tilde{b} = -\frac{c}{\delta}$*, then equation* (3.21) *becomes*

$$\psi^{+}_{SII}(u) = \frac{e^{-\frac{\lambda\eta}{c}u}}{1 + \frac{\lambda\eta}{c}C_1^{-1}} \quad u \geqslant 0,$$

where $C_1^{-1} = \int_{-\frac{c}{\delta}}^{0} e^{-\beta w}\left(\frac{\delta w}{c} + 1\right)^{\frac{\lambda}{\delta}-1} dw$ *and thus we retrieve Theorem 12 of Dassios and Embrechts (1989) for the ruin probability in the classic model with debit interest.*

Example 1 (Comparison of SII insolvency versus the classical ruin probability)**.** *The main aim of the Solvency II regulation is to provide a more prudent risk management scheme, protecting both the company and its policyholders against possible insolvency. In this paper, as can be seen in reality, we attempt to achieve this by the addition of capital injections and borrowing. Therefore, it is of interest to consider, numerically, the effect of such measures. In order to compare the insolvency probability* $\psi^{+}_{SII}(u)$*,* $u \geqslant k$ *with the classic ruin probability under exponentially distributed claim sizes, which is given by*

$$\psi(u) = \frac{1}{1+\eta} e^{-\frac{\lambda\eta}{c}u}, \quad u \geqslant 0,$$

consider the parameters $\lambda = \beta = 1$ and the positive safety loading variable $\eta = 5\%$ (typical value in the literature), which due to the net profit condition, fixes our premium rate at $c = 1.05$. We further set the debit force $\delta = 0.05$ and the fix MCR barrier $\tilde{b} = 3$, which in turn gives $b = 24$, since $b = \tilde{b} + \frac{c}{\delta}$. Table 1 (below) shows us the comparison of the classical and the SII ruin probabilities for several values of u and the SCR level k such that $u \geqslant k \geqslant b = 24$.

Furthermore, in Table 2, numerics for the required initial capital are given in the case of a fixed probability of insolvency and SCR level.

	$k = 25$		$k = 30$		$k = 50$	
u	$\psi(u)$	$\psi^+_{\mathrm{SII}}(u)$	$\psi(u)$	$\psi^+_{\mathrm{SII}}(u)$	$\psi(u)$	$\psi^+_{\mathrm{SII}}(u)$
k	0.290	0.509	0.228	6.933×10^{-3}	0.088	1.439×10^{-11}
$k+5$	0.228	0.401	0.180	5.464×10^{-3}	0.069	1.134×10^{-11}
$k+10$	0.180	0.316	0.142	4.306×10^{-3}	0.055	8.938×10^{-12}
$k+15$	0.142	0.249	0.112	3.394×10^{-3}	0.043	7.044×10^{-12}
$k+20$	0.112	0.196	0.088	2.675×10^{-3}	0.034	5.552×10^{-12}

Table 1: Classical ruin against SII insolvency probabilities, exponential claims.

	u		
$\psi^+_{\mathrm{SII}}(u)$	$k = 25$	$k = 26$	$k = 27$
0.1	59.17	47.32	31.34
0.05	73.72	61.87	45.90
0.025	88.28	76.43	60.46
0.01	107.52	95.67	79.70

Table 2: Initial capital required for varying insolvency probabilities and SCR levels

Note that in the tables above, we give only numerical results for $\psi^+_{\mathrm{SII}}(u)$ in order to be consistent with the SII framework. That is, the initial capital must be at least the value of the SCR level.

3.2 Asymptotics results for the probability of insolvency

Over the years a vast array of models have been proposed, and expressions derived, for ruin probabilities and related quantities, however explicit expressions are seldom obtained and, in fact, only some are derived even for special cases. Hence, in this subsection we will recall previously derived asymptotic expressions for the classic ruin related quantities in order to discover the behaviour of $\psi^+_{\mathrm{SII}}(u)$, $u \geqslant k$ as $u \to \infty$, which by the close relationship to the classic ruin probability, will allow us to show that the asymptotic behaviour of $\psi^+_{\mathrm{SII}}(u)$ differs by a constant factor to the the classic ruin behaviour as $u \to \infty$. We will not consider the asymptotic behaviour of $\psi^-_{\mathrm{SII}}(u)$, since $\tilde{b} < u < b$ has an upper bound at b.

Let us begin by deriving asymptotic expressions for $G_{\tilde{u}}(y)$ and $g_{\tilde{u}}(y)$. From Gerber et al. (1987), it follows that the distribution of the deficit at ruin, $G(u, y)$ satisfies the following renewal equation eqRN1

$$G(u, y) = \frac{\lambda}{c} \int_0^u G(u - x, y)\overline{F}_X(x)\, dx + \frac{\lambda}{c} \int_u^{u+y} \overline{F}_X(x)\, dx, \tag{3.25}$$

which is a *defective renewal equation* since $\frac{\lambda}{c} \int_0^\infty \overline{F}_X(x)\, dx = \frac{\lambda\mu}{c} < 1$, given that the net profit condition holds. Thus, as in Feller (1971) we can assume there exists a constant R, known as the *Lundberg exponent*, such that

$$\frac{\lambda}{c} \int_0^\infty e^{Rx}\overline{F}_X(x)\, dx = 1,$$

then, $\frac{\lambda}{c} e^{Rx}\overline{F}_X(x)$ forms a density of a proper probability function. Multiplying equation (3.25) by e^{Ru}, with R satisfying the above condition, we have eqRN2

$$e^{Ru}G(u, y) = \frac{\lambda}{c} \int_0^u e^{R(u-x)}G(u - x, y)e^{Rx}\overline{F}_X(x)\, dx + \frac{\lambda}{c} e^{Ru} \int_u^{u+y} \overline{F}_X(x)\, dx, \tag{3.26}$$

which is now in the form of a proper renewal equation. Then, direct application of the Key Renewal Theorem [see Rolski et al. (1999), Thm 6.1.11], gives that

$$\lim_{u\to\infty} e^{Ru}G(u, y) = \frac{\int_0^\infty e^{Rt} \int_t^{t+y} \overline{F}_X(x)\, dxdt}{\int_0^\infty te^{Rt}\overline{F}_X(t)\, dt}.$$

Following a similar argument [see also, Grandell (1999)], we obtain the following asymptotic expression for the classic probability of ruin

$$\lim_{u\to\infty} e^{Ru}\psi(u) = \frac{\int_0^\infty e^{Rt} \int_t^{\infty} \overline{F}_X(x)\, dxdt}{\int_0^\infty te^{Rt}\overline{F}_X(t)\, dt}.$$

Finally, since $G_u(y) = \frac{G(u,y)}{\psi(u)}$, by a similar argument as in Willmot (2002), since , we have

$$\lim_{u\to\infty} G_u(y) = \frac{\int_0^\infty e^{Rt} \int_t^{t+y} \overline{F}_X(x)\, dxdt}{\int_0^\infty e^{Rt} \int_t^{\infty} \overline{F}_X(x)\, dxdt}.$$

from which it follows, by differentiating the above equation with respect to y, that

$$\lim_{u\to\infty} g_u(y) = \frac{\int_0^\infty e^{Rt}\overline{F}_X(t + y)dt}{\int_0^\infty e^{Rt} \int_t^{\infty} \overline{F}_X(x)\, dxdt}.$$

Thus, the asymptotic behaviour of $\psi^+_{\text{SII}}(u)$ as $u \to \infty$ is given by the following Proposition.

Proposition 3. *The probability of Insolvency, $\psi^+_{SII}(u)$, behaves asymptotically as*

$$\psi^+_{SII}(u) \sim K\psi(u), \quad u \to \infty.$$

where $\psi(u)$ is the classic ruin probability and K is given by

$$K = 1 - \frac{\phi(0)\left[\int_0^\infty e^{Rt}\int_t^{t+(k-b)} \overline{F}_X(x)\,dxdt + \int_{k-b}^{k-\tilde{b}}\int_0^\infty e^{Rt}\overline{F}_X(t+y)\chi_\delta(k-y)\,dt\,dy\right]}{\frac{\mu\eta}{R}\left(1-\frac{\lambda}{c}\left(\mu F_e(k-b) + \int_{k-b}^{k-\tilde{b}}\overline{F}_X(y)\chi_\delta(k-y)\,dy\right)\right)}.$$

4 Probability characteristics of the accumulated capital injections

In order to enforce measures against insolvency, by means of capital injections, it is necessary to acquire a source of such funds. Usually, these are either; capital injections from the national government (if it is in their interest to keep the company solvent) or injections from the companies shareholders - Dickson and Waters (2004) proposed "As the shareholders benefit from the dividend income until ruin, it is reasonable to expect that the shareholders provide the initial surplus u and take care of the deficit at ruin". In extreme cases capital injections can be offered by a reinsurer, as considered by Pafumi (1998) and Nie et al. (2011), among others. Regardless from which scheme the capital injections are received, it will be prudent for the source to understand its potential liabilities, in order to manage their own portfolios. Based on such information, the primary source of funds can be compensated accordingly i.e. it allows the company to fix certain dividend levels for the shareholders based on their risk, or set a premium level to pay for a reinsurance contract.

In this section we aim to obtain the probabilistic characteristics of the accumulated capital injections up to the time of insolvency, including an expression for the moment generating function. For the latter, we show that the distribution of the accumulated capital injections up to the time of insolvency is a degenerate distribution.

4.1 Moments of the accumulated capital injections up to time of insolvency

Let the total accumulated capital injections, up to time $t \geqslant 0$, be denoted by the pure jump process $\{Z(t)\}_{t\geqslant 0}$ and consider $\mathbb{E}(Z_{u,k})$ where $Z_{u,k} = Z(T_\delta)$ is the accumulated capital injections up to the time of insolvency, given the initial capital level u. For similar reasons as the insolvency probability, $\mathbb{E}(Z_{u,k})$ can be decomposed depending on the size of the initial capital. It is therefore convenient to define $\mathbb{E}(Z_{u,k}) = \mathbb{E}(Z^+_{u,k})$ when $u \geqslant k$ and $\mathbb{E}(Z_{u,k}) = \mathbb{E}(Z^-_{u,k})$, when $\tilde{b} < u < b$. Using a similar argument as in the previous section (that is, conditioning on the amount of the first drop below the SCR barrier k), we have that $\mathbb{E}(Z^+_{u,k})$, for $u \geqslant k$, satisfies eqCInj1

$$\begin{aligned}
\mathbb{E}(Z^+_{u,k}) &= \int_0^{k-b} \left(y + \mathbb{E}(Z^+_{k,k})\right) g(\tilde{u}, y)\, dy + \int_{k-b}^{k-\tilde{b}} \left((k-b) + \mathbb{E}(Z^+_{k,k})\right) g(\tilde{u}, y)\chi_\delta(k-y)dy \\
&= \int_0^{k-b} y g(\tilde{u}, y)\, dy + (k-b) \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)dy \\
&\qquad + \mathbb{E}(Z^+_{k,k}) \left[G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)dy \right],
\end{aligned} \tag{4.1}$$

where $\chi_\delta(x)$, defined in equation (3.4) for $\tilde{b} < x < b$, has been extensively studied in the previous section. In order to complete the calculation for $\mathbb{E}(Z^+_{u,k})$, given by the above expression, we need to compute the value of $\mathbb{E}(Z^+_{u,k})$ at $u = k$, namely $\mathbb{E}(Z^+_{k,k})$, which follows immediately by setting $u = k$ in equation (4.1). Hence,

$$\begin{aligned}
\mathbb{E}(Z^+_{k,k}) &= \int_0^{k-b} y g(0, y)\, dy + (k-b) \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy \\
&\quad + \mathbb{E}(Z^+_{k,k}) \left[G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy \right],
\end{aligned}$$

from which we have that eqCInj2

$$\mathbb{E}(Z^+_{k,k}) = \frac{\int_0^{k-b} y g(0, y)\, dy + (k-b) \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy}{1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0, y)\chi_\delta(k-y)dy \right)}. \tag{4.2}$$

In order to compute $\mathbb{E}(Z^-_{u,k})$, for $\tilde{b} < u < b$, note that $\mathbb{E}(Z^-_{u,k})$ satisfies

$$\mathbb{E}(Z^-_{u,k}) = \chi_\delta(u) \left((k-b) + \mathbb{E}(Z^+_{k,k}) \right), \quad \tilde{b} < u < b,$$

with $\mathbb{E}(Z^+_{k,k})$ given by equation (4.2).

To illustrate the applicability of the results for $\mathbb{E}(Z^+_{u,k})$ and $\mathbb{E}(Z^-_{u,k})$, we give explicit expressions for the two types of the expected accumulated capital injections up to the time of insolvency, when the claim amounts are exponentially distributed. Prop5

Proposition 4. *Assume that the claim amounts follow an exponential distribution with parameter $\beta > 0$ i.e. $F(x) = 1 - e^{-\beta x}$, $x \geqslant 0$. Then, the expected accumulated capital injections, $\mathbb{E}(Z^+_{u,k})$ for $u \geqslant k$, is given by* eqExp3

$$\mathbb{E}(Z^+_{u,k}) = \frac{A_1}{\eta + C_1 \frac{c}{\lambda} e^{-\beta k}} e^{-\frac{\lambda\eta}{c}(u-k)}. \tag{4.3}$$

For $\tilde{b} < u < b$, $\mathbb{E}(Z^-_{u,k})$ is given by eqExp5

$$\mathbb{E}(Z^-_{u,k}) = \frac{A_2}{\eta + C_1 \frac{c}{\lambda} e^{-\beta k}} \int_{\tilde{b}}^{u} e^{-\beta w} \left(\frac{\delta(w-b)}{c} + 1 \right)^{\frac{\lambda}{\delta} - 1} dw, \tag{4.4}$$

where

$$A_1 = \frac{1}{\beta} \left(1 - e^{-\beta(k-b)} \right) - (k-b) C_1 \frac{c}{\lambda} e^{-\beta k}$$

and

$$A_2 = C_1 \left(\frac{1}{\beta} \left(1 - e^{-\beta(k-b)} \right) + \eta(k-b) \right).$$

Remark 6. *Proposition 4 is obtained from equation* (**??**) *and the ruin related quantities, for exponential claims, used in Section 3.1. It is not difficult to obtain an explicit expression for $\mathbb{E}\left((Z^+_{u,k})^2\right)$ and greater moments, when the claim sizes are exponentially distributed, however since computing the expressions is cumbersome, we omit the results here.*

4.2 The Distribution of the Accumulated Capital Injections up to the Time of Insolvency

In this subsection we show that the distribution of the accumulated capital injections up to the time of insolvency is a mixture of a degenerative distribution at 0 and a continuous distribution. To obtain this result, we derive the moment generating function of $Z^+_{u,k}$ and $Z^-_{u,k}$, extending the arguments of Nie et al. (2011).

First consider the case where $u = k$. Then, the probability that there is a first capital injection is; the probability that the surplus process drops, due to a claim, between k and b, which happens with probability $G(0, k-b)$; or the surplus process drops, due to a claim, between b and $\tilde{b}$ and then recovers back up to the level b before crossing $\tilde{b}$, which happens with probability $\int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy$.

Given that there is a first capital injection, the process restarts from the level k. Hence, if N denotes the number of capital injections, N has a geometric distribution with p.m.f, for $n = 0, 1, 2, \ldots$

$$\begin{aligned}\mathbb{P}(N = n) = &\left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy \right)^n \\ &\times \left(1 - \left[G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\,dy \right] \right),\end{aligned}$$

with probability generating function given by

$$\mathbb{E}(z^N) = P_N(z) = \frac{1 - \left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\, dy \right)}{1 - z\left(G(0, k-b) + \int_{k-b}^{k-\tilde{b}} g(0,y)\chi_\delta(k-y)\, dy \right)}.$$

Then, the accumulated amount of the capital injections up to the time of insolvency starting from $u = k$, namely $Z_{k,k}^+$, has a compound geometric distribution of the form

$$Z_{k,k}^+ = \sum_{i=1}^{N} V_i,$$

where $\{V_i\}_{i=1}^{\infty}$ are i.i.d random variables, denoting the size of the i-th injection, with p.d.f

$$f_V(y) = \begin{cases} \frac{g(0,y)}{G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\, dx} & 0 < y < k-b, \\ \frac{\int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\, dx}{G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\, dx} & y = k-b, \end{cases}$$

and thus the moment generating function of $Z_{k,k}^+$ (a compound geometric) can be expressed by

$$M_{Z_{k,k}^+}(z) = P_N(M_V(z)),$$

where

$$M_V(z) = \mathbb{E}(e^{zV}) = \frac{\int_0^{k-b} e^{zy} g(0,y)\, dy + e^{z(k-b)} \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\, dx}{G(0,k-b) + \int_{k-b}^{k-\tilde{b}} g(0,x)\chi_\delta(k-x)\, dx}.$$

Now, in order to find the moment generating functions of the accumulated capital injections up to the time of insolvency for general initial capital, namely $Z_{u,k}^+$ when $u \geqslant k$ and $Z_{u,k}^-$, when $\tilde{b} < u < b$, we first note that $Z_{u,k}^+$ and $Z_{u,k}^-$ are equivalent in distribution to $(Y_u^+ + Z_{k,k}^+)\mathbb{I}_{\{A^+\}}$ and $(Y_u^- + Z_{k,k}^+)\mathbb{I}_{\{A^-\}}$, respectively, where Y_u^+ is the amount of the first capital injection, starting from initial capital $u > k$, Y_u^- from initial capital $\tilde{b} < u < b$ and $\mathbb{I}_{\{\cdot\}}$ is the indicator function with respect to the event the event that a capital injections occurs from initial capital u. Note that the event that a capital injections occurs from initial capital u can be decomposed to the sub events depending the value of the initial capital and thus we denote A^+ and A^- the events that a capital injections occurs from initial capital $u > k$ and $\tilde{b} < u < b$, respectively, with probabilities

$$\mathbb{P}(A^+) = G(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u}, y)\chi_\delta(k-y)\, dy,$$

and

$$\mathbb{P}(A^-) = \chi_\delta(u).$$

Based on the above notation, for $\tilde{u} = u - k$, the density of Y_u^+ is given by

$$f_{Y_u^+}(y) = \begin{cases} \frac{g(\tilde{u},y)}{G(\tilde{u},k-b)+\int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx} & 0 < y < k-b, \\ \frac{\int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx}{G(\tilde{u},k-b)+\int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx} & y = k-b, \end{cases}$$

while Y_u^- has a probability mass function of the following form

$$\mathbb{P}(Y_u^- = i) = \begin{cases} 1, & i = k-b \\ 0 & \text{otherwise.} \end{cases}$$

Then, since Y_u^+ and $Z_{k,k}^+$ are independent, the moment generating function of $Z_{u,k}^+$ is given by

$$M_{Z_{u,k}^+}(z) = \left(M_{Y_u^+}(z) M_{Z_{k,k}^+}(z) \right) \mathbb{P}(A^+) + \mathbb{P}((A^+)^c), \tag{4.5}$$

where

$$M_{Y_u^+}(z) = \mathbb{E}(e^{zY_u^+}) = \frac{\int_0^{k-b} e^{zy} g(\tilde{u},y)\,dy + e^{z(k-b)} \int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx}{G(\tilde{u},k-b) + \int_{k-b}^{k-\tilde{b}} g(\tilde{u},x)\chi_\delta(k-x)\,dx}$$

while, following a similar argument as above, the moment generating function of $Z_{u,k}^-$ is given by

$$M_{Z_{u,k}^-}(z) = \left(M_{Y_u^-}(z) M_{Z_{k,k}^+}(z) \right) \mathbb{P}(A^-) + \mathbb{P}((A^-)^c), \tag{4.6}$$

where

$$M_{Y_u^-}(z) = \mathbb{E}(e^{zY_u^-}) = e^{z(k-b)},$$

From equations (4.5) and (4.6), it follows the following proposition.

Proposition 5. *The distribution of the accumulated capital injections up to the time of insolvency, is mixture of a degenerative distribution at* 0 *and a continuous distribution.*

5 Constant dividend barrier strategy with SII constraints

In reality the surplus of a company will not be left to grow indefinitely, and as a proportion of the profits are paid out as dividends to its shareholders. As mentioned in the previous section, the shareholders in a company are one potential source of Solvency regulation, by means of capital injections, for which they would expect financial incentives/security and therefore the consideration of dividend payments is important when analysing a firms portfolio and insolvency probabilities. Dividend strategies have been extensively studied in the risk theory literature since their introduction by De Finetti (1957), with a main focus on optimisation of the companies utility, see Avanzi (2009) and references therein for a comprehensive review.

In this section we derive an explicit expression for the insolvency probability to the risk model under the SII framework, proposed in the previous sections, with the addition of a constant dividend barrier $d \geqslant k$, such that when the surplus reaches the level d, dividends are paid continuously at rate c until a new claim appears (see Fig:2). The amended surplus process, denoted $U^Z_{\delta,d}(t)$, has dynamics

$$dU^Z_{\delta,d}(t) = \begin{cases} -dS(t), & U^Z_{\delta,d}(t) = d, \\ cdt - dS(t), & k \leqslant U^Z_{\delta,d}(t) < d, \\ \Delta Z(t), & b \leqslant U^Z_{\delta,d}(t) < k, \\ \left[c + \delta(U^Z_{\delta}(t) - b)\right] dt - dS(t), & \tilde{b} < U^Z_{\delta,d}(t) < b. \end{cases}$$

In a similar way as the model without the presence of a dividend barrier, the time to insolvency in the dividend amended model can be defined by

$$T_{\delta,d} = \inf\left\{t \geqslant 0 : U^Z_{\delta,d}(t) \leqslant \tilde{b} | U^Z_{\delta,d}(0) = u\right\}$$

and the probability of insolvency (ruin), which we denote by $\psi_{\text{SII},d}(u)$, is defined as

$$\psi_{\text{SII},d}(u) = \mathbb{P}\left(T_{\delta,d} < \infty \middle| U^Z_{\delta,d}(0) = u\right),$$

with the corresponding solvency (survival) probability defined by $\phi_{\text{SII},d}(u) = 1 - \psi_{\text{SII},d}(u)$.

We once again note that the insolvency probability $\psi_{\text{SII},d}(u)$, can be decomposed for $k \leqslant u \leqslant d$ and $\tilde{b} < u < b$, for which we define $\psi_{\text{SII},d}(u) = \psi^+_{\text{SII},d}(u)$ and $\psi_{\text{SII},d}(u) = \psi^-_{\text{SII},d}(u)$, for the two separate cases with corresponding solvency probabilities $\phi^+_{\text{SII},d}(u)$ and $\phi^-_{\text{SII},d}(u)$, respectively.

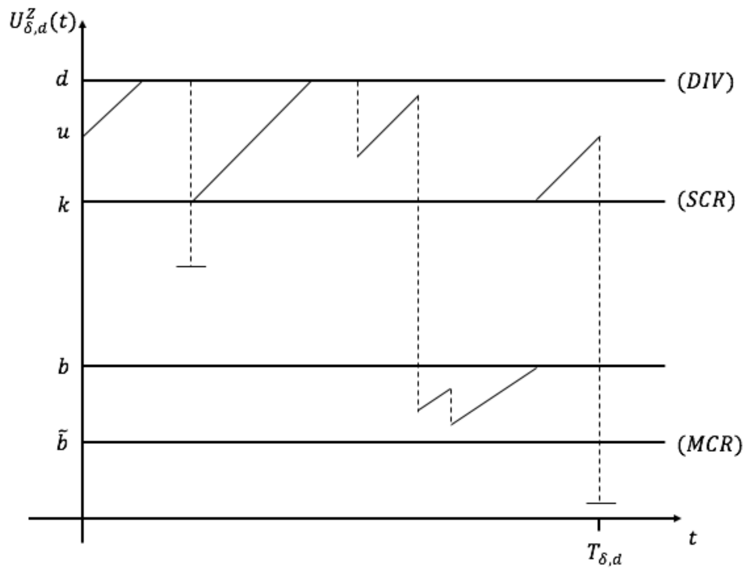

Figure 2: Example Sample Path of the Surplus Process under SII constraints with Constant Dividend Barrier

In order to derive an expression for the solvency probability $\phi^{+}_{\mathrm{SII},d}(u)$, for $k \leqslant u \leqslant d$, (or equivalently the insolvency probability $\psi^{+}_{\mathrm{SII},d}(u)$) we will need to define the first crossing time of the surplus below the SCR level k, as we did in Section 3.

Let $T_d = \inf\{t \geqslant 0 : U^Z_{\delta,d}(t) < k |\, U^Z_{\delta,d}(0) = u \geqslant k\}$ to be the first time the process down crosses the barrier k. Then, the probability of crossing the SCR level, for some $k \leqslant u \leqslant d$, can be given as

$$\xi_d(u) = \mathbb{P}(T_d < \infty | U^Z_{\delta,d}(0) = u).$$

It is evident, by a similar argument as in Section 3, that the dynamics of the surplus process $U^Z_{\delta,d}(t)$ above the SCR level are equivalent to that of the classic surplus process with a constant dividend barrier $\tilde{b} = b - k$, only (i.e. no capital injections or debit borrowing barriers). That is, for $k \leqslant U^Z_{\delta,d}(t) \leqslant d$, we have $dU^Z_{\delta,d}(t) \equiv d\widetilde{U}_{\tilde{d}}(t)$ where

$$\widetilde{U}_{\tilde{d}}(t) = \tilde{u} + ct - S(t), \qquad \widetilde{U}_{\tilde{d}}(0) = \tilde{u} \geqslant 0,$$

with dynamics

$$d\widetilde{U}_{\tilde{d}}(t) = \begin{cases} -dS(t), & \widetilde{U}_{\tilde{d}}(t) = \tilde{d}, \\ cdt - dS(t), & 0 \leqslant \widetilde{U}_{\tilde{d}}(t) < \tilde{d}. \end{cases}$$

It follows that the probability of the surplus process under the SII framework with dividends, $U^Z_{\delta,d}(t)$, for $k \leqslant u \leqslant d$, crossing the SCR level, namely $\xi_d(u)$, is simply the probability that the process $\widetilde{U}_{\tilde{d}}(t)$ crosses zero, which is given as the shifted analogue of the classical probability of ruin under a constant dividend barrier strategy, i.e. $\xi_d(u) = \psi_{\tilde{d}}(\tilde{u})$, with initial capital $0 \leqslant \tilde{u} \leqslant \tilde{d}$. It follows that the probability of never down-crossing the SCR level, for $k \leqslant u \leqslant d$, is equivalent to the shifted analogue of the classic survival probability under a constant dividend barrier i.e. $\phi_{\tilde{d}}(\tilde{u}) = 1 - \psi_{\tilde{d}}(\tilde{u}) = 1 - \xi_d(u)$. (Note that when $d = \infty$, then $T_\infty = T$ and $\xi_d(u) = \xi(u)$. That is, we return to the problem without a divided barrier as proposed in Section 3).

Now, since we have once again alluded to the connection between the probability of down crossing the SCR barrier with the shifted classic ruin probability, we further define

$$G_{\tilde{d}}(\tilde{u}, y) = \mathbb{P}\left(T_d < \infty, |\widetilde{U}_d(T_d)| \leqslant y \middle| \widetilde{U}_d(0) = \tilde{u}\right)$$

to be the distribution of the deficit below k at the time of crossing the barrier, under the constant dividend barrier constraint, with $g_{\tilde{d}}(\tilde{u}, y) = \frac{\partial}{\partial y} G_{\tilde{d}}(\tilde{u}, y)$ its corresponding density.

To obtain an expression for the insolvency probability under a constant dividend barrier strategy, let us condition on the occurrence and amount of the first drop below the SCR barrier, k. Then for $k \leqslant u \leqslant d$, the respective solvency probability $\phi^{+}_{\mathrm{SII},d}(u)$, is given by eqDB1

$$\begin{aligned}
\phi^{+}_{\mathrm{SII},d}(u) &= \phi_{\tilde{d}}(\tilde{u}) + \int_0^{k-b} g_{\tilde{d}}(\tilde{u}, y)\phi^{+}_{\mathrm{SII},d}(k)\, dy + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(\tilde{u}, y)\phi^{-}_{\mathrm{SII},d}(k-y)\, dy \\
&= \phi_{\tilde{d}}(\tilde{u}) + G_{\tilde{d}}(\tilde{u}, k-b)\phi^{+}_{\mathrm{SII},d}(k) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(\tilde{u}, y)\phi^{-}_{\mathrm{SII},d}(k-y)\, dy. \qquad (5.1)
\end{aligned}$$

For $\tilde{b} < u < b$, we have eqDB2

$$\phi^{-}_{\mathrm{SII},d}(u) = \chi_\delta(u)\phi^{+}_{\mathrm{SII},d}(k), \qquad (5.2)$$

where $\chi_\delta(u)$ is the probability of hitting the upper barrier b before the lower barrier $\tilde{b}$, in a debit environment, as studied in the previous sections. We point out that the function $\chi_\delta(u)$ is unaffected by the addition of the dividend barrier and therefore the integro-differential equation given in Proposition 2 still holds, along with the corresponding boundary conditions. Following similar algebraic arguments as in Section 3 we obtain the following Theorem. ThmDB1

Theorem 3. *For $k \leqslant u \leqslant d$,the probability of insolvency under a constant dividend barrier strategy, $\psi^{+}_{SII,d}(u)$, satisfies* eqDB36

$$\psi^{+}_{SII,d}(u) = \psi_{\tilde{d}}(\tilde{u}) - \frac{\phi_{\tilde{d}}(0)\left[G_{\tilde{d}}(\tilde{u}, k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(\tilde{u}, y)\chi_\delta(k-y)\, dy\right]}{1 - \left(G_{\tilde{d}}(0, k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(0, y)\chi_\delta(k-y)\, dy\right)}. \qquad (5.3)$$

For $\tilde{b} < u < b$, $\psi^{-}_{SII,d}(u)$ is given by eqDB4

$$\psi^{-}_{SII,d}(u) = 1 - \frac{\phi_{\tilde{d}}(0)\chi_\delta(u)}{1 - \left(G_{\tilde{d}}(0, k-b) + \int_{k-b}^{k-\tilde{b}} g_{\tilde{d}}(0, y)\chi_\delta(k-y)\, dy\right)}. \qquad (5.4)$$

Rem10

Remark 7. *Similarly to Remark 2, we point out that from equations* (5.3) *and* (5.4)*, that the two types of insolvency probabilities for the risk model under SII constraint with the addition of a constant dividend barrier, are given in terms of the (shifted) ruin probability and deficit of the classical risk model with constant dividend barrier, as well as the probability of exiting between two barriers. Thus,* $\psi^{+}_{SII,d}(\cdot)$ *and* $\psi^{-}_{SII,d}(\cdot)$ *can be calculated by employing known results, with respect to* $G_d(\cdot,\cdot)$ *and* $\psi_d(\cdot)$ *(see Lin et al. (2003), among others), whilst the latter exiting probability,* $\chi_\delta(u)$*, has been extensively studied in Section 3.*

In more details, Lin et al. (2003), show that the well known Gerber-Shiu function - for which the ruin probability and deficit at ruin are special cases (for details see Gerber and Shiu (1998)) - under a constant divided barrier strategy, denoted by $m_d(u)$, satisfies an integro-differential equation, from which the general solution can be expressed as a linear combination of the corresponding Gerber-Shiu function without the presence of dividends and a secondary function $v(u)$. That is, the Gerber-Shiu function under a constant dividend barrier strategy, namely $m_d(u)$, with initial capital $0 \leqslant u \leqslant d$, can be expressed as eqDVG

$$m_d(u) = m_\infty(u) - \frac{m'_\infty(d)}{v'(d)} v(u), \quad 0 \leqslant u \leqslant d, \tag{5.5}$$

where $m_\infty(u)$ is the classic Gerber-Shiu function without dividend constraints and $v(u)$ is a function satisfying a homogenous integro-differential equation, from which the general solution is given by

$$v(u) = \frac{1-\Psi(u)}{1-\Psi(0)},$$

for some auxiliary function $\Psi(u)$, the details of which are not needed for this paper. However, we point out that when the Gerber-Shiu function is reduced to the special cases of the ruin probability or the deficit at ruin, for which equation (5.5) holds, the auxiliary function above is equivalent to the classic ruin function i.e. $\Psi(u) = \psi(u)$.

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